

Fermion Masses and Flavor Mixing in A Supersymmetric $SO(10)$ Model

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Abstract: we study fermion masses and flavor mixing in a supersymmetric $SO(10)$ model, where **10**, **120** and **$\overline{126}$** Higgs multiplets have Yukawa couplings with matter multiplets and give masses to quarks and leptons through the breaking chain of a Pati-Salam group. This brings about that, at the GUT energy scale, the lepton mass matrices are related to the quark ones via several breaking parameters, and the small neutrino masses arise from a Type II see-saw mechanism. When evolving renormalization group equations for the fermion mass matrices from the GUT scale to the electroweak scale, in a specific parameter scenario, we show that the model can elegantly accommodate all observed values of masses and mixing for the quarks and leptons, especially, it's predictions for the bi-large mixing in the leptonic sector are very well in agreement with the current neutrino experimental data.

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I. Introduction

The flavor problem about fermion masses and mixing has been an open question in particle physics [1]. In the Standard Model (SM), the fermion masses and mixing angles are completely arbitrary and neutrinos are massless, but recent experiments uniquely specify that neutrinos have small masses and bi-large leptonic mixing angles [2]. The present global analysis gives at 90% C.L. [3]

$$\begin{aligned} \Delta m_{21}^2 &\approx 7.1 \times 10^{-5} \text{ eV}^2, \quad |\Delta m_{32}^2| \approx (1.3 - 3.0) \times 10^{-3} \text{ eV}^2, \\ \tan^2 \theta_{12} &\approx 0.4, \quad \sin^2 2\theta_{23} > 0.92, \quad \sin \theta_{13} < 0.23, \end{aligned} \quad (1)$$

which are distinctly different from large masses and small mixing angles in the quark sector. Incorporating the diverse values of fermion masses and mixing into a theory with the small number of parameters is a large challenge for theoretical particle physicists. There have been many ideas proposed for the purpose [4], among others, supersymmetric (SUSY) models based on the grand unified theory (GUT) such as SUSY $SO(10)$ models are theoretically well-motivated extensions of the SM and have received growing attention [5].

SUSY $SO(10)$ models provide the most natural framework for generation of neutrino mass and realization of see-saw mechanism [6], in addition, the fermion mass hierarchies are also understood very well by appealing to additional family symmetries [7]. However, details of symmetry breaking chains and Higgs field structures play a crucial role in specifying the model and determining the fermion masses and mixing angles [5, 8]. One can generally obtain the correct quantum number for the SM particle content and implement the see-saw mechanism through the breaking chain of either the Pati-Salam group $SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$ (G_{422}) or $SU(5) \otimes U(1)$ (G_{51}). The former breaks the left-right symmetry $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ (G_{3221}) at the $B-L$ energy scale and gives the right-handed neutrino masses by means of the renormalizable couplings of the fermion fields in **16** spinor representations with the Higgs multiplets in $\overline{\mathbf{126}}$ representation, in which the R -parity is automatically conserved at all energy scales, whereas the latter makes use of the nonrenormalizable couplings of the matter spinor fields with the **16** Higgs multiplets to generate neutrino masses. The different symmetry breaking chains give rise to different mass relations between the quark sector and the lepton sector. However, a successful GUT model is required to naturally account for the difference of masses and mixing between the quarks and the leptons. The minimal SUSY $SO(10)$ model [9], where only one **10** and one $\overline{\mathbf{126}}$ Higgs multiplets have symmetric Yukawa couplings with matter multiplets, seems difficult to consistently incorporate the realistic neutrino oscillation parameters. It was recently realized that there are possibilities for reproducing observed fermion masses and mixing by considering new Higgs multiplets [10].

In this work, we extend the minimal SUSY $SO(10)$ model by introducing anti-symmetric couplings of one **120** Higgs multiplets with the matter fields. Supposing the G_{422} breaking chain, the left-right symmetry G_{3221} is broken by the G_{422} component $(\mathbf{10}, \mathbf{1}, \mathbf{3})$ contained in the $\overline{\mathbf{126}}$ Higgs multiplets and large Majorana masses

are given to the right-handed neutrinos. Below the $B-L$ energy scale, this model is described as the minimal supersymmetric see-saw standard model with superfield couplings of the left-handed lepton doublets with the $SU(2)_L$ Higgs triplets, which are from the $(\overline{\mathbf{10}}, \mathbf{3}, \mathbf{1})$ component of $\overline{\mathbf{126}}$ and can give small Majorana masses to the left-handed neutrinos. The two Higgs doublets in the minimal supersymmetric standard model (MSSM) are linear combinations of the $SU(2)_L$ doublet components from different $SO(10)$ representations of Higgses, which all contribute to electroweak symmetry breaking and give Dirac masses to all the fermions. The effective Majorana masses for the light left-handed neutrinos are generated through the Type II see-saw mechanism [11]. Because of receiving contributions from different components of the $\mathbf{120}$ Higgs multiplets, the mass relation between the up-quark sector and the Dirac neutrino sector is different from the mass relation between the down-quark sector and the charged lepton sector. As we will see below, the characteristic structure of Higgs fields and symmetry breaking superpotential can lead to fit completely all the observed data of masses and mixing for the quarks and the leptons.

The remainder of this paper is organized as follows. In Section II we outline the model and derive the GUT relation among the fermion mass matrices, and then the renormalization group equations (RGEs) of the mass matrices are introduced. In Sec. III, a detailed numerical analysis of the masses and mixing angles of the neutrinos is given in a specific parameter scenario satisfying the experimental constraints. Sec. IV is devoted to conclusions.

II. Model, Mass Matrices and Renormalization Group Evolution

We consider the SUSY model based on the GUT gauge group $SO(10)$. The matter fields Ψ^{16} in one $\mathbf{16}$ spinor representation of the $SO(10)$ group contain all the quarks and leptons as well as the right-handed neutrino of each generation. Higgs fields which can couple to spinor fermions at the renormalizable level include only the H^{10} , H^{120} , and $H^{\overline{126}}$ multiplets of the $\mathbf{10}$, $\mathbf{120}$, and $\overline{\mathbf{126}}$ representations under $SO(10)$, respectively. The gauge invariant Yukawa couplings superpotential are such as

$$W_{SO(10)} = y_{ij}^{10} \Psi_i^{16} H^{10} \Psi_j^{16} + y_{ij}^{\overline{126}} \Psi_i^{16} H^{\overline{126}} \Psi_j^{16} + y_{ij}^{120} \Psi_i^{16} H^{120} \Psi_j^{16}, \quad (2)$$

where i, j are generation indices. By virtue of the gauge symmetry, the Yukawa couplings, y_{ij}^{10} and $y_{ij}^{\overline{126}}$, are symmetric 3×3 matrices, while y_{ij}^{120} is an antisymmetric one. In our model, all the Yukawa couplings are assumed to be real to keep the number of free parameters minimum.

We suppose that the GUT gauge symmetry descends to $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ (G_{321}) of the SM through two intermediate symmetries which are orderly the Pati-Salam subgroup G_{422} and the left-right symmetry subgroup G_{3221} . At first, the $SO(10)$ symmetry is broken down to G_{3221} by developing non-vanishing vacuum expectation values (VEVs) $\langle \mathbf{54} \rangle + \langle \mathbf{45} \rangle$ of two Higgs multiplets in Higgs superpotential.

The subsequent breaking to G_{321} is achieved by the $(\mathbf{10}, \mathbf{1}, \mathbf{3})$ component VEV of $H^{\overline{126}}$ in eq.(2), which is a singlet under G_{321} . In the framework of G_{321} , accordingly the Yukawa superpotential are rewritten as

$$\begin{aligned}
W_{G_{321}} = & \hat{Q}_i \left[y_{ij}^{10} H_{(1,2,2)_u}^{10} - \frac{1}{3} y_{ij}^{\overline{126}} H_{(15,2,2)_u}^{\overline{126}} + y_{ij}^{120} \left(H_{(1,2,2)_u}^{120} + \frac{1}{3} H_{(15,2,2)_u}^{120} \right) \right] \hat{u}_j^c \\
& + \hat{Q}_i \left[y_{ij}^{10} H_{(1,2,2)_d}^{10} - \frac{1}{3} y_{ij}^{\overline{126}} H_{(15,2,2)_d}^{\overline{126}} + y_{ij}^{120} \left(-H_{(1,2,2)_d}^{120} + \frac{1}{3} H_{(15,2,2)_d}^{120} \right) \right] \hat{d}_j^c \\
& + \hat{L}_i \left[y_{ij}^{10} H_{(1,2,2)_d}^{10} + y_{ij}^{\overline{126}} H_{(15,2,2)_d}^{\overline{126}} + y_{ij}^{120} \left(-H_{(1,2,2)_d}^{120} - H_{(15,2,2)_d}^{120} \right) \right] \hat{e}_j^c \\
& + \hat{L}_i \left[y_{ij}^{10} H_{(1,2,2)_u}^{10} + y_{ij}^{\overline{126}} H_{(15,2,2)_u}^{\overline{126}} + y_{ij}^{120} \left(H_{(1,2,2)_u}^{120} - H_{(15,2,2)_u}^{120} \right) \right] \hat{\nu}_j^c \\
& + \hat{L}_i y_{ij}^{\overline{126}} H_{(\overline{10},3,1)}^{\overline{126}} \hat{L}_j + \hat{\nu}_i^c y_{ij}^{\overline{126}} \left\langle H_{(10,1,3)}^{\overline{126}} \right\rangle \hat{\nu}_j^c, \tag{3}
\end{aligned}$$

where as usually the notations $\hat{Q}, \hat{L}, \hat{u}, \hat{e}$, etc. denote the chiral superfields for the quarks and leptons, and the subscripts of various Higgs superfields show that they respectively originated from corresponding representations of G_{422} , the lowest indices u, d stand for belonging to the up-type and down-type $SU(2)_L$ Higgs doublet of G_{321} , respectively. Note that a Clebsch-Gordon coefficient (-3) is generated in the lepton sectors when the components $(\mathbf{15}, \mathbf{2}, \mathbf{2})$ of G_{422} are involved in the superpotential, which plays a crucial role in obtaining the Georgi-Jarlskog relations [12]. The last second couplings will yield the left-handed neutrino Majorana masses after the neutral component of the $SU(2)_L$ Higgs triplet $H_{(\overline{10},3,1)}^{\overline{126}}$ develops a VEV as small as the neutrino masses measured in experiments. The last term gives the right-handed neutrino Majorana masses, moreover, the VEV $\left\langle H_{(10,1,3)}^{\overline{126}} \right\rangle$ should be close to the GUT energy scale so as to successfully implement the see-saw mechanism. At last, the electroweak symmetry breaking is accomplished by the electroweak scale VEVs of neutral components of these Higgs doublets in eq.(3), through which all the fermions acquire Dirac masses.

From the point of view of phenomenology, omitting the heavy Higgs fields decoupling from the low energy (here we assume that a doublet-doublet Higgs mass splitting is realized), the two light Higgs doublets in the MSSM, whose VEVs are v_u, v_d , are realistically identified with some appropriate linear combinations of the above up-type and down-type Higgs doublets, respectively. This corresponds that the up-type Higgs doublet VEVs are in proportion to v_u , while the down-type ones are relative to v_d . To sum up, the various Higgs field VEVs in eq.(3) are described by

$$\begin{aligned}
\left\langle H_{(1,2,2)_{u/d}}^{10} \right\rangle &= c_{u/d}^{(1)} v_{u/d}, \quad \left\langle H_{(15,2,2)_{u/d}}^{\overline{126}} \right\rangle = c_{u/d}^{(2)} v_{u/d}, \\
\left\langle H_{(1,2,2)_{u/d}}^{120} \right\rangle &= c_{u/d}^{(3)} v_{u/d}, \quad \left\langle H_{(15,2,2)_{u/d}}^{120} \right\rangle = c_{u/d}^{(4)} v_{u/d}, \\
\left\langle H_{(\overline{10},3,1)}^{\overline{126}} \right\rangle &= c_L \frac{v^2}{M_G}, \quad \left\langle H_{(10,1,3)}^{\overline{126}} \right\rangle = c_R M_G, \tag{4}
\end{aligned}$$

where $v_u = \sin \beta v$, $v_d = \cos \beta v$, and $v = 174$ GeV is the VEV of the Higgs field in the SM, M_G is the GUT energy scale. The coefficients $c_{u/d}^{(k)}$ ($k = 1, \dots, 4$) arising from the mixing of the Higgs doublets are in general complex, through which CP -violating phases are introduced into the fermion mass matrices. The parameters c_L, c_R are positive, by which the mass scales of the left-handed and right-handed neutrino are related to the electroweak and GUT energy scale v, M_G .

After the above symmetry breaking, the lagrangian relevant to the fermion masses is now obtained as follows

$$\begin{aligned}
-\mathcal{L}_{\text{mass}} = & \bar{u}_{Li} M_{ij}^u u_{Rj} + \bar{d}_{Li} M_{ij}^d d_{Rj} + \bar{e}_{Li} M_{ij}^e e_{Rj} + \bar{\nu}_{Li} M_{ij}^\nu \nu_{Rj} \\
& + \frac{1}{2} \bar{\nu}_{Li} M_{ij}^L \nu_{Lj} + \frac{1}{2} \bar{\nu}_{Ri} M_{ij}^R \nu_{Rj} + h.c. ,
\end{aligned} \tag{5}$$

where the Dirac mass matrices for the quarks and leptons, as well as the Majorana mass matrices for the left-handed and right-handed neutrinos are, respectively, given by

$$\begin{aligned}
M_{ij}^u &= v \sin \beta \left[c_u^{(1)} y_{ij}^{10} - \frac{1}{3} c_u^{(2)} y_{ij}^{\overline{126}} + \left(c_u^{(3)} + \frac{1}{3} c_u^{(4)} \right) y_{ij}^{120} \right] , \\
M_{ij}^d &= v \cos \beta \left[c_d^{(1)} y_{ij}^{10} - \frac{1}{3} c_d^{(2)} y_{ij}^{\overline{126}} + \left(-c_d^{(3)} + \frac{1}{3} c_d^{(4)} \right) y_{ij}^{120} \right] , \\
M_{ij}^e &= v \cos \beta \left[c_d^{(1)} y_{ij}^{10} + c_d^{(2)} y_{ij}^{\overline{126}} + \left(-c_d^{(3)} - c_d^{(4)} \right) y_{ij}^{120} \right] , \\
M_{ij}^\nu &= v \sin \beta \left[c_u^{(1)} y_{ij}^{10} + c_u^{(2)} y_{ij}^{\overline{126}} + \left(c_u^{(3)} - c_u^{(4)} \right) y_{ij}^{120} \right] , \\
M_{ij}^L &= \frac{2v^2}{M_G} c_L y_{ij}^{\overline{126}} , \\
M_{ij}^R &= 2M_G c_R y_{ij}^{\overline{126}} .
\end{aligned} \tag{6}$$

After integrating out the heavy right-handed neutrino fields decoupling from theory below the B - L breaking scale, the effective Majorana masses of the light left-handed neutrinos are realized by the type II see-saw mechanism as

$$M_{\text{eff}}^L = M^L - M^\nu (M^R)^{-1} (M^\nu)^T . \tag{7}$$

All the fermion mass matrices in eq.(6) are characterized by two energy scales v, M_G , three positive breaking vacuum parameters $\tan \beta, c_L, c_R$, three real Yukawa coupling matrices $y_{ij}^{10}, y_{ij}^{\overline{126}}, y_{ij}^{120}$, and eight complex mixing coefficients c_u^k, c_d^k , but these parameters are not independent each other.

Owing to only the three Yukawa coupling matrices, the fermion mass matrices in eq.(6) are, at the GUT scale, correlated each other. We define the following

notations as

$$\begin{aligned}
M_{S,A}^f &= \frac{M^f \pm (M^f)^T}{2} \quad (f = u, d, e, \nu), \\
r_l &= \frac{c_u^{(l)}}{c_d^{(l)}} \tan \beta \quad (l = 1, 2, 3), \quad r_4 = \frac{c_u^{(4)}}{c_u^{(3)}}, \quad r_5 = \frac{c_d^{(4)}}{c_d^{(3)}}, \\
r_L &= \frac{c_L}{c_d^{(2)} \cos \beta}, \quad r_R = \frac{c_R}{c_d^{(2)} \cos \beta}.
\end{aligned} \tag{8}$$

The matrices $M_{S,A}^f$ denotes the symmetric and antisymmetric parts of the fermion Dirac mass matrices, respectively. The dimensionless parameters r_l, r_4, r_5 are virtually some ratios of the Higgs doublet VEVs in eq.(4). By virtue of the special definitions, it is easy to derive the GUT relations among the quark and lepton mass matrices as follows

$$\begin{aligned}
M_A^u &= \frac{r_3(3+r_4)}{-3+r_5} M_A^d, \\
M_S^e &= \frac{4}{r_1-r_2} M_S^u - \frac{3r_1+r_2}{r_1-r_2} M_S^d, \quad M_A^e = \frac{-3(1+r_5)}{-3+r_5} M_A^d, \\
M_S^\nu &= \frac{r_1+3r_2}{r_1-r_2} M_S^u - \frac{4r_1r_2}{r_1-r_2} M_S^d, \quad M_A^\nu = \frac{3r_3(1-r_4)}{-3+r_5} M_A^d, \\
M^L &= 6r_L \left(\frac{v}{M_G} \right) \left[\frac{1}{r_1-r_2} M_S^u - \frac{r_1}{r_1-r_2} M_S^d \right], \\
M^R &= 6r_R \left(\frac{M_G}{v} \right) \left[\frac{1}{r_1-r_2} M_S^u - \frac{r_1}{r_1-r_2} M_S^d \right].
\end{aligned} \tag{9}$$

The symmetric parts of the fermion mass matrices are expressed as linear combinations of M_S^u and M_S^d , while the antisymmetric parts of them are linearly dependent on only M_A^d . Various coefficients of the linear combinations are determined by the seven parameters $r_1, \dots, r_5, r_L, r_R$, through which the lepton mass matrices are predicted by the quark mass matrices.

In the following discussion, we consider the model in a specific scenario. First, adopting a generation basis in which the up-type quark mass matrix M^u is real and diagonal. Secondly, assuming that the down-type quark mass matrix M^d is Hermitian. Namely, the quark mass matrices are described as

$$M^u = \text{diag}(m_u^0, m_c^0, m_t^0), \quad M^d = U_{CKM}^0 \text{diag}(m_d^0, m_s^0, m_b^0) (U_{CKM}^0)^\dagger, \tag{10}$$

where U_{CKM}^0 is the CKM matrix with three mixing angles and one CP -phase in the quark sector [13], the superscript 0 means that the quark masses and mixing are evaluated at the GUT scale (hereafter as such). As a result, it is very easy to know from eq.(6) that the relation $c_u^{(4)} = -3c_u^{(3)}$, and the coefficients $c_{u/d}^{(1)}, c_{u/d}^{(2)}$ are real, while $c_d^{(3)}, c_d^{(4)}$ are pure imaginary. Lastly, making a supposition that the coefficients

$c_u^{(3)}, c_u^{(4)}$ are also real. To collect these together, we can draw a conclusion that all the charged fermion mass matrices in eq.(6), M^u, M^d, M^e , are Hermitian, while all the neutral ones, M^ν, M^L, M^R , are real. The CP -violating sources come from only the antisymmetric parts in the down-type quark and charged lepton sectors. Furthermore, It can be inferred by eq.(8) that $r_4 = -3$, and r_1, r_2, r_5, r_L, r_R are real (moreover, r_L and r_R have the same sign) but r_3 is a pure imaginary. In terms of eq.(9), now this six parameters in addition to the ten values of the quark masses and mixing in eq.(10) determine fully all the mass matrices in the lepton sector.

On account of the Hermite matrix M^e satisfying the constraint equations

$$\begin{aligned} \text{Tr} M^e &= m_e^0 + m_\mu^0 + m_\tau^0, \quad \text{Det} M^e = m_e^0 \cdot m_\mu^0 \cdot m_\tau^0, \\ \text{Tr} [(M^e)^2] &= (m_e^0)^2 + (m_\mu^0)^2 + (m_\tau^0)^2, \end{aligned} \quad (11)$$

the three parameters r_1, r_2, r_5 can actually be solved out via inputting the charged lepton mass eigenvalues. Although one can find two sets of real solutions, whose parts involved in r_1, r_2 are the same except the parts r_5 are different, the two M^e matrices determined respectively by them are only a transpose each other. Without loss of generality, we only consider the solution whose r_5 is relatively larger in the following analysis. Therefore, the three masses of the charged leptons fix completely the parameters r_1, r_2, r_5 , sequentially determine the charged lepton mass matrix by eq.(9).

Now the left parameters r_L, r_R, r_3 are restricted through fitting the masses and mixing in the neutrino sector. In terms of eq.(7), the effective left-handed neutrino Majorana mass matrix M_{eff}^L is a real symmetric, but only the two mass-squared differences are known from eq.(1), so we consider the following simultaneous equations

$$\begin{aligned} \text{Tr} M_{\text{eff}}^L &= m_1^0 + m_2^0 + m_3^0, \quad \text{Det} M_{\text{eff}}^L = m_1^0 \cdot m_2^0 \cdot m_3^0, \\ \text{Tr} [(M_{\text{eff}}^L)^2] &= (m_1^0)^2 + (m_2^0)^2 + (m_3^0)^2, \\ \Delta^0 m_{21}^2 &= (m_2^0)^2 - (m_1^0)^2, \quad \Delta^0 m_{32}^2 = (m_3^0)^2 - (m_2^0)^2. \end{aligned} \quad (12)$$

Taking $\Delta^0 m_{21}^2, \Delta^0 m_{32}^2, r_3$ as input parameters, the values of $m_1^0, m_2^0, m_3^0, r_L, r_R$ are solved out. The normal (or inverted) hierarchy corresponds to $\Delta^0 m_{32}^2 > 0$ (or < 0). Note that four sets of real solutions can be found for eq.(12), but only two sets satisfy conditions that r_L, r_R have the same sign. In fact, the two sets of solutions are only contrary sign. Putting them into eq.(9) and using eq.(7), they actually give the same matrix M_{eff}^L , so one can adopt either of them.

To summarize the above scenario, there are in all sixteen independent input parameters evaluated at the GUT scale. They are six quark masses, three angles and one CP -phase in the quark CKM matrix, three charged lepton masses, two neutrino mass-squared differences and one free parameter r_3 . Providing these parameter values, all the fermion mass matrices at the GUT scale are entirely determined. Furthermore, running the fermion mass matrices from the GUT scale to the electroweak

scale according to the following introduced RGEs, we can compare our results with the experimental data of the fermion masses and mixing at the electroweak scale.

Below the $B-L$ scale, the right-handed neutrinos are decoupled by the see-saw mechanism. The model particle spectrum are identical to ones of the MSSM with the effective left-handed Majorana neutrinos. We introduce the Yukawa coupling squared matrices for the charged fermions as follows

$$\begin{aligned} S_u &= \frac{1 + (\tan \beta)^{-2}}{v^2} (M^u)^\dagger M^u, \quad S_d = \frac{1 + (\tan \beta)^2}{v^2} (M^d)^\dagger M^d, \\ S_e &= \frac{1 + (\tan \beta)^2}{v^2} (M^e)^\dagger M^e, \end{aligned} \quad (13)$$

and then the one-loop close RGEs for these Yukawa coupling squared matrices, the effective neutrino mass matrix and the gauge coupling constants of G_{321} are given by [14]

$$\frac{d\alpha_i(\chi)}{d\chi} = \frac{b_i}{2\pi} \alpha_i^2, \quad (i = 1, 2, 3) \quad (14)$$

$$\frac{dS_f(\chi)}{d\chi} = \frac{1}{16\pi^2} (S_f K_f + K_f S_f), \quad (f = u, d, e) \quad (15)$$

$$\frac{dM_{\text{eff}}^L(\chi)}{d\chi} = \frac{1}{16\pi^2} [M_{\text{eff}}^L K_\nu + (K_\nu)^T M_{\text{eff}}^L], \quad (16)$$

with

$$\begin{aligned} K_u &= 3S_u + S_d + \left[\text{Tr}(3S_u) - 4\pi \left(\frac{13}{15}\alpha_1 + 3\alpha_2 + \frac{16}{3}\alpha_3 \right) \right] I, \\ K_d &= S_u + 3S_d + \left[\text{Tr}(3S_d + S_e) - 4\pi \left(\frac{7}{15}\alpha_1 + 3\alpha_2 + \frac{16}{3}\alpha_3 \right) \right] I, \\ K_e &= 3S_e + \left[\text{Tr}(3S_d + S_e) - 4\pi \left(\frac{9}{5}\alpha_1 + 3\alpha_2 \right) \right] I, \\ K_\nu &= S_e + \left[\text{Tr}(3S_u) - 4\pi \left(\frac{3}{5}\alpha_1 + 3\alpha_2 \right) \right] I, \end{aligned} \quad (17)$$

where $\chi = \ln(Q/M_G)$, $b_i = (33/5, 1, -3)$, and I is a 3×3 unit matrix. Note that the eq.(14) and eq.(15) close on themselves, in addition, we neglect the running effects for the right-handed Majorana neutrinos since their mass scale is close to the GUT scale. For given $\tan \beta$, the previous calculated mass matrices and one unified gauge coupling constant are taken as the input values at the GUT energy scale $Q_{GUT} = M_G$, consequently, we can solve the above RGEs numerically and achieve the values of $\alpha_i(\chi)$, $S_f(\chi)$, $M_{\text{eff}}^L(\chi)$ at the electroweak energy scale $Q_{\text{weak}} = M_Z$. The electroweak scale fermion mass eigenvalues are subsequently obtained by diagonalizing the Yukawa coupling squared matrices and the effective neutrino mass

matrix as follows

$$\begin{aligned}
U_u S_u(\chi_w) U_u^\dagger &= \frac{1 + (\tan \beta)^{-2}}{v^2} \text{diag} (m_u^2(\chi_w), m_c^2(\chi_w), m_t^2(\chi_w)), \\
U_d S_d(\chi_w) U_d^\dagger &= \frac{1 + (\tan \beta)^2}{v^2} \text{diag} (m_d^2(\chi_w), m_s^2(\chi_w), m_b^2(\chi_w)), \\
U_e S_e(\chi_w) U_e^\dagger &= \frac{1 + (\tan \beta)^2}{v^2} \text{diag} (m_e^2(\chi_w), m_\mu^2(\chi_w), m_\tau^2(\chi_w)), \\
U_\nu M_{\text{eff}}^L(\chi_w) U_\nu^T &= \text{diag} (m_1(\chi_w), m_2(\chi_w), m_3(\chi_w)),
\end{aligned} \tag{18}$$

where $\chi_w = \ln(M_Z/M_G)$ denotes that the fermion masses and mixing are evaluated at the electroweak scale. Accordingly, the quark and lepton mixing matrices are given by [13, 15]

$$U_u U_d^\dagger = U_{CKM}^q(\chi_w), \quad U_e U_\nu^\dagger = U_{CKM}^l(\chi_w) \text{diag} (e^{i\beta_1}, e^{i\beta_2}, 0), \tag{19}$$

where β_1, β_2 are two Majorana phases in the lepton mixing matrix. Finally, the mixing angles and CP -violating phases in the unitary matrices $U_{CKM}^{q,l}(\chi_w)$ are worked out by the standard parameterization in ref. [16].

III. Numerical Results

In this section, we present numerical results of our model. First of all, we fix the electroweak scale, the GUT scale and the unified gauge coupling constants such as

$$M_Z = 91.2 \text{ GeV}, \quad M_G = 3.1 \times 10^{16} \text{ GeV}, \quad \alpha_1(0) = \alpha_2(0) = \alpha_3(0) = 0.0408. \tag{20}$$

According to eq.(14), three gauge coupling constants at the electroweak scale are found to be

$$\alpha_1(\chi_w) \approx 0.0168, \quad \alpha_2(\chi_w) \approx 0.0335, \quad \alpha_3(\chi_w) \approx 0.1172. \tag{21}$$

These are in accordance with the current experimental measures very well [16].

Secondly, the Higgs VEV v is also fixed to $v = 174 \text{ GeV}$, and the two representative values of $\tan \beta$ are taken as $\tan \beta = 10$ and $\tan \beta = 30$. In the case $\tan \beta = 10(30)$ (Note, hereafter as such, the values in the parenthesis are corresponding to the case $\tan \beta = 30$), we input the quark and charged lepton masses as well as the quark mixing angles and CP -violating phase at the GUT scale as follows (in GeV)

$$\begin{aligned}
m_u^0 &= 0.00106 (0.00107), \quad m_c^0 = 0.307 (0.312), \quad m_t^0 = 136 (143), \\
m_d^0 &= 0.00125 (0.00135), \quad m_s^0 = 0.0265 (0.0286), \quad m_b^0 = 1.02 (1.18), \\
m_e^0 &= 0.000324 (0.00035), \quad m_\mu^0 = 0.0685 (0.074), \quad m_\tau^0 = 1.17 (1.32), \\
s_{12}^0 &= 0.2229 (0.2229), \quad s_{23}^0 = 0.0348 (0.0339), \quad s_{13}^0 = 0.003 (0.003), \quad \delta_{13}^0 = 59^\circ (59^\circ).
\end{aligned} \tag{22}$$

After using eq.(14) and eq.(15), the equivalent values of the above masses and mixing at the electroweak scale are solved out such that (in GeV)

$$\begin{aligned}
m_u(\chi_w) &\approx 0.00233, \quad m_c(\chi_w) \approx 0.677, \quad m_t(\chi_w) \approx 181, \\
m_d(\chi_w) &\approx 0.00439, \quad m_s(\chi_w) \approx 0.0929, \quad m_b(\chi_w) \approx 3.01, \\
m_e(\chi_w) &\approx 0.000487, \quad m_\mu(\chi_w) \approx 0.103, \quad m_\tau(\chi_w) \approx 1.75, \\
s_{12}^q(\chi_w) &\approx 0.2229, \quad s_{23}^q(\chi_w) \approx 0.0412, \quad s_{13}^q(\chi_w) \approx 0.0036, \quad \delta_{13}^q(\chi_w) \approx 59^\circ.
\end{aligned} \tag{23}$$

They are completely consistent with the current status of the quark and charged lepton masses and the CKM matrix at the M_Z scale [16].

Lastly, we take the input values of the neutrino mass-squared differences at the GUT scale and the parameter r_3 as (remember that r_3 is a pure imaginary)

$$\begin{aligned}
\Delta^0 m_{21}^2 &= 1.39 (1.50) \times 10^{-4} \text{ eV}^2, \quad \Delta^0 m_{32}^2 = 3.9 (4.3) \times 10^{-3} \text{ eV}^2, \\
r_3 &= 759i (1105i).
\end{aligned} \tag{24}$$

Here, we only consider the case of the normal hierarchy, namely $\Delta^0 m_{32}^2 > 0$. For the case of the inverted hierarchy, a similar analysis can also be performed. After the RGEs of eq.(14)–(16) running simultaneously, the output values of the masses and mixing for the effective left-handed neutrinos at the electroweak scale are found as follows (in eV)

$$\begin{aligned}
m_1 &\approx 0.0084 (0.0050), \quad m_2 \approx 0.0119 (0.0098), \quad m_3 \approx 0.0463 (0.0459), \\
\Delta m_{21}^2 &\approx 7.1 (7.1) \times 10^{-5}, \quad \Delta m_{32}^2 \approx 2.0 (2.0) \times 10^{-3}, \\
\sin^2 2\theta_{12} &\approx 0.816 (0.816), \quad \sin^2 2\theta_{23} \approx 0.960 (0.948), \quad \sin^2 2\theta_{13} \approx 0.154 (0.082), \\
\tan^2 \theta_{12} &\approx 0.40 (0.40), \quad \sin \theta_{13} \approx 0.20 (0.14), \\
\delta_{13}^l &\approx -0.041\pi (-0.039\pi), \quad \beta_1 \approx 1.045\pi (0.045\pi), \quad \beta_2 \approx 0.495\pi (1.495\pi).
\end{aligned} \tag{25}$$

The above results are excellently in agreement with the recent neutrino oscillation experimental data in eq.(1). The three neutrino masses are all within experimental limits. The values of m_2 and m_3 are close to 0.01 eV and 0.046 eV, respectively, but m_1 varies with $\tan \beta$ about range (0.005 – 0.008) eV. The two mass-squared differences and the three mixing angles are all at the center values of the allowed regions. The leptonic CP -violating phase δ_{13}^l is approximate to -0.04π ($\approx -7.2^\circ$), which is small very much in comparison with the quark one. One of the two Majorana phases is about π (or 0), while the other is $\pi/2$ (or $3\pi/2$) or so. In addition, the right-handed neutrino masses are obtained straightforward by diagonalizing the mass matrix M^R in eq.(9) such that (in GeV)

$$\begin{aligned}
M_1 &\approx 9.1 \times 10^{12} (1.1 \times 10^{14}), \quad M_2 \approx 1.7 \times 10^{14} (3.4 \times 10^{14}), \\
M_3 &\approx 1.7 \times 10^{15} (4.8 \times 10^{15}).
\end{aligned} \tag{26}$$

They are close to the GUT scale except that the M_1 value is slightly low for the case $\tan \beta = 10$, which are just expected in the previous sections.

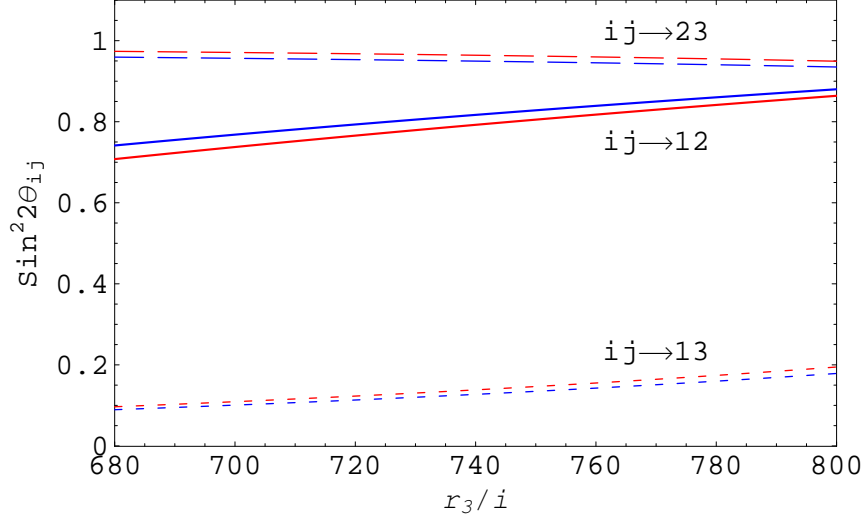


FIG. 1. For $\tan\beta = 10$, the three leptonic mixing angles as functions of the parameter r_3/i . The red and blue curves correspond to $\Delta m_{32}^2 \approx 2.0 \times 10^{-3} \text{ eV}^2$ and $3.0 \times 10^{-3} \text{ eV}^2$, respectively, and they all hold $\Delta m_{12}^2 \approx 7.1 \times 10^{-5} \text{ eV}^2$.

Now we illustrate the numerical analysis in detail. In Figs. 1 and 2 the three leptonic mixing angles $\sin^2 2\theta_{ij}$ ($ij = 12, 23, 13$) are shown as functions of the model parameter r_3/i for the case $\tan\beta = 10$ and $\tan\beta = 30$, respectively. In the case $\tan\beta = 10$ (30), we fix the input $\Delta^0 m_{21}^2 = 1.39$ (1.50) $\times 10^{-4} \text{ eV}^2$ at the GUT scale so that $\Delta m_{21}^2 \approx 7.1 \times 10^{-5} \text{ eV}^2$ is always held at the electroweak scale. If the value of $\Delta^0 m_{32}^2$ at the GUT scale is taken as $\Delta^0 m_{32}^2 = 3.9$ (4.3) $\times 10^{-3} \text{ eV}^2$, which leads to $\Delta m_{32}^2 \approx 2.0 \times 10^{-3} \text{ eV}^2$ at the electroweak scale, this case is depicted by the red curves in each figure. If $\Delta^0 m_{32}^2 = 5.9$ (6.4) $\times 10^{-3} \text{ eV}^2$, the value of Δm_{32}^2 is accordingly increased to $\Delta m_{32}^2 \approx 3.0 \times 10^{-3} \text{ eV}^2$, this case corresponds to the blue curves in each figure. The other input parameters are given by eq.(20) and (22).

The graphs show explicitly the bi-large mixing characteristic of the lepton flavor mixing, namely, a very large solar angle $\sin^2 2\theta_{12} \sim (0.7 - 0.9)$, a nearly maximal atmospheric angle $\sin^2 2\theta_{23} \sim (0.9 - 1)$, and a very small CHOOZ angle $\sin^2 2\theta_{13} \sim (0.05 - 0.2)$. As the value of r_3/i is increased in the displayed range, the atmospheric angle $\sin^2 2\theta_{23}$ is almost horizontal and always hold the near maximum. The solar and CHOOZ angles go up slowly, but they still lie in the experimental bounds. The best fit for the experimental data is at $r_3/i \sim 760$ (1100) for the case $\tan\beta = 10$ (30). In addition, when the electroweak scale Δm_{23}^2 varies from $2.0 \times 10^{-3} \text{ eV}^2$ to $3.0 \times 10^{-3} \text{ eV}^2$, the plots only shift finely without changing the overall trends. Finally, it is worth to point out that r_3/i varying has almost no effect on the values of Δm_{12}^2 and Δm_{32}^2 at the electroweak scale, which are principally determined by the GUT scale $\Delta^0 m_{12}^2$ and $\Delta^0 m_{32}^2$, respectively. In short, our model predict naturally the bi-large mixing structure, moreover, the three mixing angles and two mass-squared differences of the neutrino oscillations are realized simultaneously for certain

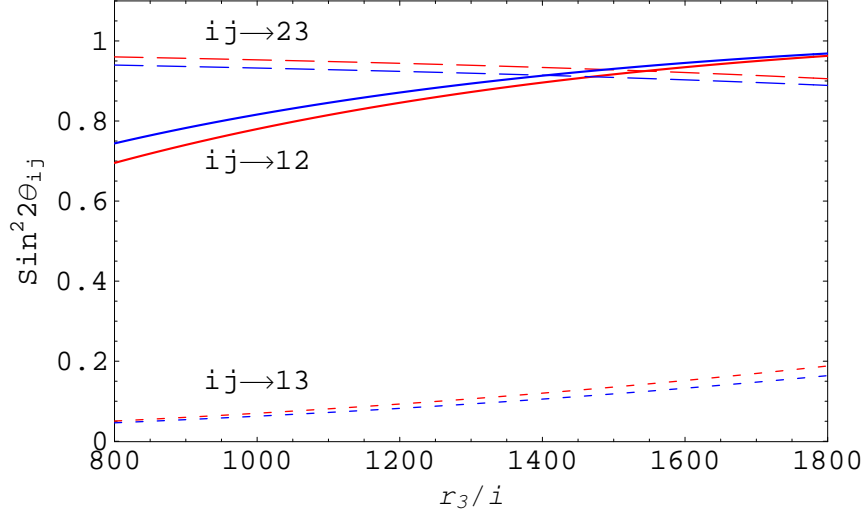


FIG. 2. For $\tan\beta = 30$, the three leptonic mixing angles as functions of the parameter r_3/i . The red and blue curves correspond to $\Delta m_{32}^2 \approx 2.0 \times 10^{-3} \text{ eV}^2$ and $3.0 \times 10^{-3} \text{ eV}^2$, respectively, and they all hold $\Delta m_{12}^2 \approx 7.1 \times 10^{-5} \text{ eV}^2$.

parameter regions.

IV. Conclusions

In summary, we have discussed the fermion masses and flavor mixing within the framework of the SUSY $SO(10)$ model. In the GUT model, we introduce the anti-symmetric couplings of the **120** Higgs with the matter fields besides the symmetric couplings of the **10** and **126** Higgs in the minimal SUSY $SO(10)$ model. The new Yukawa couplings and the left-right symmetry breaking chain not only bring about that the quark and lepton mass matrices are closely related to each other, but also lead to elegant explanation for the difference between the large flavor mixing for the leptons and the small flavor mixing for the quarks. After the renormalization group evolution from the GUT scale to the electroweak scale, all the current experimental data for the fermion masses and mixing as well as the gauge couplings are reproduced correctly in our model parameter scenario. In particular, the bi-large mixing structure for the leptons is naturally arisen in certain regions of the parameter space, moreover, the solar and atmospheric mass-squared differences for the neutrinos are also accommodated simultaneously. All the nontrivial results indicate clearly that the SUSY $SO(10)$ GUT model with the **120** Higgs couplings is worth investigating deeply for understanding the mystery of the fermion flavor.

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